Kalkulus (1240033)

Bab V Turunan

Juwairiah, S.Si,M.T
(juwai_riah@yahoo.com)

Sub Pokok Bahasan

- Turunan Fungsi Invers
- Turunan Fungsi Siklometri
- Turunan Fungsi Eksponen
- Turunan Fungsi Logaritma
- Aturan Rantai
- Aturan Rantai Bersusun

Kompetensi Khusus

Mahasiswa mampu menyelesaikan berbagai turunan fungsi

TURUNAN FUNGSI INVERS

Teorema :

Jika y = f (x) mempunyai invers x = g (y), maka : $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$

Contoh : 1. Tentukan $\frac{dy}{dx}$ jika diketahui $y = \sqrt{x}$

Jawab : $y = \sqrt{x}$ mempunyai invers : $x = y^2$

Maka : $\frac{dx}{dy} = 2y$

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{2y} = \frac{1}{2\sqrt{x}}$$

TURUNAN FUNGSI SIKLOMETRI

$$y = arc\sin x \rightarrow y' = \frac{1}{\sqrt{1-x^2}}$$

$$y = arc\cos x \rightarrow y' = \frac{-1}{\sqrt{1-x^2}}$$

$$y = arc\ tg\ x \rightarrow y' = \frac{1}{1+x^2}$$

$$y = arc\ cotg\ x \rightarrow y' = \frac{-1}{1+x^2}$$

$$y = arc\ sec\ x \rightarrow y' = \frac{1}{x\sqrt{x^2-1}}$$

$$y = arc\ cosec\ x \rightarrow y' = \frac{-1}{x\sqrt{x^2-1}}$$

Contoh :

1) $y = arc \sin x$

$$x = \sin x$$

$$x' = \frac{dx}{dy} = \cos x$$

$$= \sqrt{1 - x^2}$$

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{\sqrt{1-x^2}}$$

2) $y = arc \sin 2x$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-(2x)^2}} . 2 = \frac{2}{\sqrt{1-4x^2}}$$

3) $y = arc\ tg\ x^2$

$$\frac{dy}{dx} = y' = \frac{1}{1+(x^2)^2} . 2x = \frac{2x}{1+x^4}$$

Bilangan Alam

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} \cdot b^k \qquad \binom{n}{k} = \frac{n!}{(n-k)!\,k!} = \text{kombinasi}$$

$$= a^n + n.a^{n-1} \cdot b + \frac{n.(n-1)}{2!} a^{n-2} \cdot b^2 + \frac{n(n-1)(n-2)}{3!} a^{n-3} \cdot b^3 + \cdots + b^n$$

$$\left(1+\frac{1}{n}\right)^n \Rightarrow a = 1 \text{ dan } b = \frac{1}{n}$$

$$= 1^n + \cancel{n}.1^{n-1} \cdot \frac{1}{\cancel{n}} + \frac{n.(n-1)}{2!} 1^{n-2} \left(\frac{1}{n}\right)^2 + \cdots + \left(\frac{1}{n}\right)^n$$

$$= 1 + 1 + \frac{1}{2!} \frac{n(n-1)}{n.n} + \frac{1}{3!} \frac{(n)}{n} \frac{(n-1)}{n} \frac{(n-2)}{n} + \cdots + \left(\frac{1}{n}\right)^n$$

$$= 2 + \frac{1}{2!} \cdot 1 \cdot \left(1-\frac{1}{n}\right) + \frac{1}{3!} 1 \cdot \left(1-\frac{1}{n}\right) \cdot \left(1-\frac{2}{n}\right) + \cdots + \left(\frac{1}{n}\right)^n$$

$$\Rightarrow \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = 2+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}+\cdots$$

$$= 2{,}718281828459\cdots$$

$$= e$$

$$\therefore \boxed{\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e}$$

$$u = \frac{1}{x} \Rightarrow x = \frac{1}{u}.$$

$$x\to\infty \Rightarrow u\to 0.$$

$$\boxed{\lim_{u\to 0}(1+u)^{1/u} = e}$$

$$* \lim_{x \to 0} \frac{a^x - 1}{x}$$

$$u = a^x - 1 \rightarrow a^x = u + 1$$

$$x \to 0 \Rightarrow u \to 0$$

$$\log a^x = \log(u+1)$$

$$x \cdot \log a = \log(u+1)$$

$$x = {}^a\log(u+1)$$

$$= \lim_{u \to 0} \frac{u}{{}^a\log(u+1)}$$

$$= \lim_{u \to 0} \frac{1}{\frac{1}{u} \cdot {}^a\log(u+1)} = \frac{1}{\lim_{u \to 0} {}^a\log(u+1)^{1/u}}$$

$$= \frac{1}{{}^a\log \lim_{u \to 0} (u+1)^{1/u}} = \frac{1}{{}^a\log e} = \frac{\log a}{\log e} = {}^e\log a = \ln a$$

$$\therefore \boxed{\lim_{x \to 0} \frac{a^x - 1}{x} = \ln a}$$

TURUNAN FUNGSI EKSPONEN

$$f(x) = a^x$$

$$y' = \frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{a^{x+h} - a^x}{h}$$

$$= \lim_{h \to 0} \frac{a^x (a^h - 1)}{h} = a^x \cdot \lim_{h \to 0} \frac{(a^h - 1)}{h}$$

$$= a^x \cdot \ln a$$

$$\boxed{f(x) = e^x \Rightarrow \frac{dy}{dx} = e^x \cdot \ln e = e^x}$$

TURUNAN FUNGSI LOGARITMA

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$$f(x) = y = {}^{a}\log x \rightarrow \quad x = a^{y} \quad \text{(invers)}$$

$$\frac{dx}{dy} = a^{y} \cdot \ln a = x \ln a$$

$$\frac{dy}{dx} = \frac{1}{x \cdot \ln a}$$

$$\boxed{f(x) = y = \ln x = {}^{e}\log x \rightarrow \frac{dy}{dx} = \frac{1}{x \cdot \ln e} = \frac{1}{x}}$$

TURUNAN FUNGSI EKSPONEN & LOGARITMA

- $y = f(x) = e^x \rightarrow f'(x) = dy/dx = e^x$
- $y = f(x) = \ln x \rightarrow f'(x) = dy/dx = 1/x$

ATURAN RANTAI

Misalkan y = f(u) dan u = g(x), maka:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Contoh :

1) Diketahui $y = (x^3+3)^5$. Hitunglah $\frac{dy}{dx}$

substitusi ; $u = x^3+3 \rightarrow \frac{du}{dx} = 2x$

$y = u^5 \rightarrow \frac{dy}{du} = 5\,u^4$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 5u^4 . 2x = 5(x^2+3)^4 2x$$

$$= 10x.(x^2+3)^4$$

2) y = $\sin(x^3 - 3x)$. Hitunglah $\frac{dy}{dx}$

Subst : y = sin u dengan u = $x^3 - 3x$

$$\frac{dy}{du} = \cos u \qquad \frac{du}{dx} = 3x^2 - 3$$

$$\frac{dy}{dx} = \frac{dy}{du} . \frac{du}{dx} = \cos u . (3x^2 - 3)$$

= [cos ($x^2 - 3$)]. $(3x^2 - 3)$

Berdasarkan aturan rantai didapat rumus perluasan :

1) $y = [f(x)]^n \rightarrow y' = n[f(x)]^{n-1}.f'(x)$

2) $y = \sin[f(x)] \rightarrow y' = \cos[f(x)].f'(x)$

3) $y = \cos[f(x)] \rightarrow y' = -\sin[f(x)].f'(x)$

4) $y = e^{f(x)} \rightarrow y' = e^{f(x)}.f'(x)$

5) $y = \ln[f(x)] \rightarrow y' = \frac{1}{f(x)}.f'(x)$

Contoh :

$y = (x^3 + 2x)^4 \rightarrow y' = 4.(x^3 + 2x)^3.(3x^2 + 2)$

$y = \sin(x^3 + 2) \rightarrow y' = \cos(x^3 + 2).3x^2$

$y = \cos 4x \rightarrow y' = -\sin(4x).4 = -4.\sin(4x)$

$$y = e^{3x-1} \rightarrow y' = e^{3x-1}.3 = 3e^{3x-1}$$

$$y = \ln(x^2+1) \rightarrow y' = \frac{1}{x^2+1}.2x = \frac{2x}{x^2+1}$$

Soal – soal :

1) $y = \frac{1}{(2x^5-7)^3}$

2) $y = (x^2 - x)^{10}$

3) $y = cos^3(x^2+1)$

4) $y = \sin(3x^2 + 11x)$

5) $y = e^{(2x^5-7)}$

6) $y = cos^2 x$

7) $y = \frac{x^2 - x}{x^2}$

8) $y = (2x - 1)^3 . (x^2 - 3)^2$

9) $y = sin \frac{(3x-1)}{2x+5}$

10) $y = (\frac{x^2-1}{x+4})^4$

ATURAN RANTAI BERSUSUN

Jika y = f (u) dan u = g (v), v = h (x)

Maka : $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx}$

Contoh :

1)Carilah dy/dx dari $y = \sin^3(4x)$

Jawab : $y = u^3$ dan u = sin v, v = 4x

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx}$$

$= 3u^2 \cdot \cos v \cdot 4$

$= 3.\sin^2(4x).\cos(4x).4$

$= 12 \sin^2(4x).\cos 4x$

Aturan Rantai Bersusun (Langsung):

- Turunkan dari fungsi terluar sampai fungsi terdalam→ dikalikan

Soal

Tentukan dy/dx dari:

1. $y = sin^6(2x^4 - 7x^2)$
2. $y = \sin(2x^4 - 7x^2)^6$
3. $y = cos^6(2x^4 - 7x^2)$
4. $y = \tan(2x^4 - 7x^2)^6$
5. $y = sec^8\left(\frac{3x+2}{4x-3}\right)$
6. $y = \ln(x^3 + 4x)^{15}$
7. $y = ln^{15}(x^3 + 4x)$
8. $y = e^{\csc(x^2-6x^3)}$
9. $y = ln^8\left(\frac{x^2+2}{x^2-3}\right)$

Referensi

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